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# Correspondence 

# A Group of Linear Spectral Transformations for Two-Dimensional Digital Filters <br> JOSÉ M. COSTA AND ANASTASIOS N. VENETSANOPOULOS 


#### Abstract

There are eight linear transformations of the spectral plane which map the frequency axes onto themselves. These transformations can be used to change the pass and stop regions of a two-dimensional digital filter. A stable realization is assured by transforming the data rather than the system transfer function.


## I. Introduction

In two-dimensional recursive filter design, stability and causality are major requirements. In many cases these requirements impose severe constraints on the filter frequency response which can be attained. When these constraints limit the geometrical shape of the pass and stop regions of the two-dimensional digital filter, something can be done to change that shape. Indeed, instead of modifying the transfer function of the filter, which in most cases would lead to an unstable filter, transformations of the input and output data may result in a stable system with the desired transfer function.
Some of the transformations that we consider have already been suggested in the literature. Indeed, this work was motivated by the zero-phase technique outlined in [1]. Also, data rotations have been used in the past [2] to design stable recursive filters with circularly symmetric magnitude response. The purpose of this correspondence is to give a unified presentation with emphasis on the stable realization of the filters by equivalent data transformations.
Other spectral transformations for two-dimensional digital filters based on a different approach have been studied elsewhere [3].

## II. A Group of Linear Spectral Transformations

Suppose that we are given a stable and causal ${ }^{1}$ twodimensional recursive filter with frequency response $H\left(\omega_{1}, \omega_{2}\right)$. Consider all the linear transformations of the spectral plane which map the spectral axes onto themselves. There are eight possible such transformations and they have the algebraic structure of a finite group ${ }^{2}$ under the operation of multiplication [4]. These transformations and their effect on the frequency response of the digital filter are the following (see Fig. 1):
(a) $H\left(\omega_{1}, \omega_{2}\right)$ identity
(b) $H\left(\omega_{2},-\omega_{1}\right)$. clockwise $90^{\circ}$ rotation
(c) $H\left(-\omega_{1},-\omega_{2}\right)$ clockwise $180^{\circ}$ rotation
(d) $H\left(-\omega_{2}, \omega_{1}\right)$ clockwise $270^{\circ}$ rotation
(e) $H\left(-\omega_{1}, \omega_{2}\right)$ vertical mirror image
(f) $H\left(-\omega_{2},-\omega_{1}\right)$ transpose with respect to principal diagonal

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${ }^{1}$ Causality for two-dimensional recursive filters is defined in [5].
2 In the algebraic literature this group is referred to as the dihedral group of order 8 .


Fig. 1. (a) -(h) Group of linear spectral transformations.
(g) $H\left(\omega_{1},-\omega_{2}\right)$ horizontal mirror image
(h) $H\left(\omega_{2}, \omega_{1}\right)$ transpose with respect to secondary diagonal.
These transformations could equivalently be defined in the $z_{1}, z_{2}$ domain by complex conjugating and/or interchanging the complex variables $z_{1}$ and $z_{2}$ in the filter transfer function, but the transformations are easier to visualize in the frequency domain.

Nevertheless, among the eight transformed filters only $H\left(\omega_{1}, \omega_{2}\right)$ and $H\left(\omega_{2}, \omega_{1}\right)$ are causal and stable, because of the symmetry of the stability condition [1, theorem 1]. All other transformed filters are noncausal and have different stability conditions which are incompatible with those of $H\left(\omega_{1}, \omega_{2}\right)$ and $H\left(\omega_{2}, \omega_{1}\right)$ [5, theorem 2], [6]; and, therefore, they are unstable. Also, it should be noticed that if the impulse response of the filter $H\left(\omega_{1}, \omega_{2}\right)$ is real, then both $H\left(\omega_{1}, \omega_{2}\right)$ and $H\left(-\omega_{1},-\omega_{2}\right)$ have the same magnitude response, which is symmetric with respect to the origin, and opposite phase responses. For this reason, from $H\left(\omega_{1}, \omega_{2}\right)$ only four different magnitude responses can be derived by the transformations shown. The other transfer functions are useful for filters with zero-phase response. This is achieved by combining in series or in parallel two filters with the same magnitude response and opposite phase response [1].

There are two alternatives to realize the unstable filters. The first is to filter the data in a different manner [1] by using an algorithm recursing in another direction. Fig. 2 shows the direction of recursion, sense of recursion, and the starting point on the input data for each of the transformed filters. A horizontal arrow denotes a recursion by rows and a vertical arrow denotes a recursion by columns. The head of the arrow points the sense of recursion and the base of the arrow shows the starting point.
The second alternative is to transform the data [2]. Indeed, it is easily shown that the system described by (1) in the frequency domain is equivalent to the system in (2). This follows directly from the fact that the application of a transformation distributes over pointwise multiplication.

$$
\begin{align*}
Y\left(\omega_{1}, \omega_{2}\right) & =\left[T H\left(\omega_{1}, \omega_{2}\right)\right] \cdot X\left(\omega_{1}, \omega_{2}\right)  \tag{1}\\
& =T\left\{H\left(\omega_{1}, \omega_{2}\right) \cdot\left[T^{-1} X\left(\omega_{1}, \omega_{2}\right)\right]\right\} \tag{2}
\end{align*}
$$

where $X\left(\omega_{1}, \omega_{2}\right)$ and $Y\left(\omega_{1}, \omega_{2}\right)$ are the Fourier transforms of the input and output of the system, respectively, and the


Fig. 2. (a)-(h) Realization of a transformed filter by changing the direction and sense of recursion of the filter algorithm.
transfer function of the system is $H\left(\omega_{1}, \omega_{2}\right)$ affected by the transformation $T$, which can be any of those eight described previously. In (2) the inverse transform is applied to the input data, before filtering with $H\left(\omega_{1}, \omega_{2}\right)$, and the transform itself to the output data, after filtering. This process is illustrated in the block diagram of Fig. 3. This sequence of operations gives the desired transfer function and guarantees stability because the recursive filtering is done with $H\left(\omega_{1}, \omega_{2}\right)$ which is stable by definition. Data transformations do not affect stability because if a filter is stable it will be stable no matter what the (bounded) input is. Also, the data transformations in Fig. 3 affect neither the linearity nor the space-invariance of the system.
The system (2) shown in Fig. 3 is best realized in the space domain. The transformations are easily applied to the data which are given in the form of a matrix and they can be done in place if the matrix is square. Here we use the property that a orthogonal change of coordinates in the space domain results in the same change of coordinates in the frequency domain [7].
Systems like that in Fig. 3 can be cascaded to obtain a system with zero phase or other useful symmetries. In this case it should be noted that whenever two transformations are contiguous they can be combined together in a single transformation because the set of transformations form a group under the operation of multiplication. A multiplication table for these transformations is given in [4].

One final remark about the use of these techniques is that not only a filter must be stable, but also the filter output array must be large enough, so that the effect of truncating the filter output is negligible.

## III. Conclusions

A group of linear spectral transformations for two-dimensional digital filters has been presented. There are eight transformations which permit changing the pass and stop regions of a two-dimensional digital filter. Since in most cases the transformed filter would be unstable, an equivalent stable system has been proposed which transforms the data rather than the system transfer function. The principal applications of these transformations exist when filters with zero phase or other useful symmetries are required.

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Fig. 3. Realization of a transformed filter by equivalent data transformations.
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## A Correct Proof of Huang's Theorem on Stability

## DANIEL L. DAVIS


#### Abstract

A correct proof of Huang's theorem on the stability of two-dimensional causal recursive digital filters is developed using a maximum modulus theorem for algebraic functions.


## I. Introduction

Huang [4] has stated the following result concerning the stability of two-dimensional causal, recursive digital filters.

## Theorem 1

A causal filter with a $z$ transform

$$
H\left(z_{1}, z_{2}\right)=A\left(z_{1}, z_{2}\right) / B\left(z_{1}, z_{2}\right)
$$

where $A$ and $B$ are polynomials is stable if and only if:

1) the map of $\partial d_{1}=\left\{z_{1} \| z_{1} \mid=1\right\}$ in the $z_{2}$ plane, according to the relation $B\left(z_{1}, z_{2}\right)=0$, lies outside $d_{2}=$ $\left\{z_{2}| | z_{2} \mid \leqslant 1\right\} ;$ and
2) no point in $d_{1}=\left\{z_{1} \| z_{1} \mid \leqslant 1\right\}$ maps into the point $z_{2}=0$ by the relation $B\left(z_{1}, z_{2}\right)=0$.

The method used by Huang to prove this result is to attempt to show that conditions 1) and 2) are equivalent to the condition

$$
\begin{equation*}
B\left(z_{1}, z_{2}\right) \neq 0 \quad\left|z_{1}\right| \leqslant 1,\left|z_{2}\right| \leqslant 1 \tag{1.1}
\end{equation*}
$$

which is known to be a necessary and sufficient condition for stability [4]. In a subsequent paper, Jury and Anderson [1] observe correctly that 1) and 2) are equivalent to

$$
\begin{array}{ll}
\left.1^{\prime}\right) & B\left(z_{1}, 0\right) \neq 0
\end{array} \text { if }\left|z_{1}\right| \leqslant 1 .
$$

Thus, in order to prove Theorem 1 , it is necessary to show that conditions (1.2) are equivalent to condition (1.1). Clearly condition (1.1) implies conditions (1.2). Moreover, it is not difficult to see that conditions (1.2) imply condition (1.1) if they imply the condition

$$
\begin{equation*}
B\left(z_{1}, z_{2}\right) \neq 0 \quad \text { if }\left|z_{1}\right|<1,\left|z_{2}\right| \leqslant 1, \tag{1.3}
\end{equation*}
$$

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